What to do today ?

Part I. Introduction and Preparation

Part II. Inference under Multivariate Normal Distribution (Chp 4-7)

II.1 Multivariate Normal Distribution (Chp 4)

II.2 Inferences on Mean Vector (Chp 5)

II.3 Comparisons of Several Mean Vectors (Chp 6.1-4, 6-7)

II.4 Multivariate Linear Regression (Chp 7)

- **II.4.1 Classical Linear Regression (Chp7.1-3)**
- **II.4.2 Linear Regression Based Inference (Chp7.4-5)**
- *II.4.3 Model Checking (Chp7.6)*
- *II.4.4 Multivariate Multiple Regression (Chp7.7)*

I.4.1 Random Vectors and Matrices (Chp 2.5) - review

- **random vector (multivariate random variable).**

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{pmatrix} = (X_1, X_2, \ldots, X_p)'$$

  is a p-dim random vector if $X_1, \ldots, X_p$ are r.v.s.

- **distribution.** The cdf of $\mathbf{X}$ is the joint cumulative distribution function (joint cdf) of $X_1, \ldots, X_p$: for $\mathbf{x} = (x_1, x_2, \ldots, x_p)'$,

$$F(\mathbf{x}) = P(\mathbf{X} \leq \mathbf{x}) = P(X_1 \leq x_1, \ldots, X_p \leq x_p).$$

  - Iff $X_1, \ldots, X_p$ are multually independent from each other,

$$F(\mathbf{x}) = P(X_1 \leq x_1) \ldots P(X_p \leq x_p) = \prod_{j=1}^{p} F_{X_j}(x_j),$$

    the product of $X_1, \ldots, X_p$'s (*marginal*) cdfs.

Part I.4.2 Mean Vectors and Covariance Matrices (Chp 2.6) - review

- **expectation (population mean).** p-dim random vector $\mathbf{X}$'s expectation:

$$E(\mathbf{X}) = \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_p) \end{pmatrix},$$

denoted by $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)'$.

- **(population) variance.** p-dim r.v. $\mathbf{X}$'s variance:

$$V(\mathbf{X}) = E\left[\mathbf{X} - E(\mathbf{X})\right]\left[\mathbf{X} - E(\mathbf{X})\right]' = E(\mathbf{X}\mathbf{X}') - (E\mathbf{X})(E\mathbf{X})',$$

denoted by

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2p} \\ \vdots & \vdots & \ldots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \ldots & \sigma_{pp} \end{pmatrix}$$

with $\sigma_{ij} = \sigma_{ji} = Cov(X_i, X_j)$.

II.1.1A. Multivariate normal distribution

Definition. A p-dim r.v. $\mathbf{X}$ has a *normal* distribution if its pdf

$$f(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{\sqrt{|2\pi\boldsymbol{\Sigma}|}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}, \ -\infty < \mathbf{x} < \infty,$$

where $\boldsymbol{\Sigma}$ is positive definite. Denote it by $\mathbf{X} \sim MN_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$.

- If $\mathbf{X} \sim MN_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$, $E(\mathbf{X}) = \boldsymbol{\mu}$ and $V(\mathbf{X}) = \boldsymbol{\Sigma}$.
- $MN_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$: a family of (multivariate) distributions.
  - e.g. $MN_p(\mathbf{0},\mathbf{I})$, *the standard (multivariate) normal distribution*.

II.2 Inferences on Mean Vectors (Chp 5, Chp 6)

What statistical models have we considered?

- Consider r.v. $X \sim N(\mu, \sigma^2)$. with iid observations $X_1, \ldots, X_n$
  - Inference target: μ. Point estimates? Interval estimates?
  - Hypothesis tests? e.g. $H_0 : \mu = \mu_0$. What if σ is known/unknown?
- Consider r.v. $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with iid observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$
  - Inference target: $\boldsymbol{\mu}$. Point estimates? Interval estimates?
  - Hypothesis tests? e.g. $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$. What if $\boldsymbol{\Sigma}$ is known/unknown?
- Consider g populations $\mathbf{X}_1, \ldots, \mathbf{X}_g$. Suppose $\mathbf{X}_j \sim MN_p(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$ for $j = 1, \ldots, g$
  - Inference target: linear functions of the mean vectors $\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_g$. Point estimates? Interval estimates?
  - Hypothesis tests? e.g.
    - equality of mean vectors across groups,
    - no group effect in certain directions.
  - Inference on $\boldsymbol{\Sigma}_1, ..., \boldsymbol{\Sigma}_g$?

II.2 Inferences on Mean Vectors (Chp 5, Chp 6)

What statistical models have we considered?

- Consider r.v. $X \sim N(\mu, \sigma^2)$. with iid observations $X_1, \ldots, X_n$

$$x_r = \mu + \epsilon_r, \qquad \epsilon_r \overset{\text{iid}}{\sim} N(0, \sigma^2), \quad r = 1, \ldots, n.$$

- Consider r.v. $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with iid observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$

$$\mathbf{X}_r = \boldsymbol{\mu} + \boldsymbol{\epsilon}_r, \qquad \boldsymbol{\epsilon}_r \overset{\text{iid}}{\sim} MN_p(\mathbf{0}, \boldsymbol{\Sigma}), \quad r = 1, \ldots, n.$$

- Consider g populations $\mathbf{X}_1, \ldots, \mathbf{X}_g$. Suppose $\mathbf{X}_j \sim MN_p(\boldsymbol{\mu}_j, \boldsymbol{\Sigma}_j)$ for $j = 1, \ldots, g$

$$\mathbf{X}_{lr} = \boldsymbol{\mu}_l + \boldsymbol{\epsilon}_{lr} = \boldsymbol{\mu} + \boldsymbol{\tau}_l + \boldsymbol{\epsilon}_{lr}, \qquad \boldsymbol{\epsilon}_{lr} \overset{\text{iid}}{\sim} MN_p(\mathbf{0}, \boldsymbol{\Sigma}).$$

$$\mathbf{X}_{lkr} = \boldsymbol{\mu}_{lk} + \boldsymbol{\epsilon}_{lkr} = \boldsymbol{\mu} + \boldsymbol{\tau}_l + \boldsymbol{\beta}_k + \boldsymbol{\gamma}_{lk} + \boldsymbol{\epsilon}_{lkr}, \qquad \boldsymbol{\epsilon}_{lkr} \overset{\text{iid}}{\sim} MN_p(\mathbf{0}, \boldsymbol{\Sigma}).$$

How about to study $Y|X$?

II.4.1 Classical Linear Regression (Chp7.1-3):
II.4.1A. Introduction

- Statistical problems have we considered?
  - One-sample problems
  - Two-sample problems
  - Multi-sample problems
  - Regression problems
    - How Y depends on X?
      To establish the relationship, to predict for Y by X:
      Consider $E\big(Y|X=x\big)=h(x)$.
      e.g. *Simple Linear Regression Model.* $h(x)=\beta_0+\beta_1 x$

    - How Y depends on $X_1,\ldots,X_k$?
      To establish the relationship, to predict for Y by $X_1,\ldots,X_k$:
      Consider $E\big(Y|X_1=x_1,\ldots,X_k=x_k\big)=h(x_1,\ldots,x_k)$
      e.g. *Multiple Linear Regression Model.*
      $h(x_1,\ldots,x_k)=\beta_0+\beta_1 x_1+\ldots+\beta_k x_k$.
      (to be reviewed/studied as **II.4.1** (Chp 7.1-3), **II.4.2** (Chp 7.4-5), and **II.4.3** (Chp 7.6))

    - How $\mathbf{Y}$ depends on $X_1,\ldots,X_k$?
      Multivariate Multiple Regression to be studied as **II.4.4** based on Chp7.7

II.4.1A. Introduction. An Example

What is the regression?
Definition: *the regression of Y on X is*

$$E\left(Y|X = x\right)$$

- Example: X, Y are heights of Father, (adult) Son.
- Classic data. In R: *data("GaltonFamilies")*.
- Sons about 0.1 inch taller than fathers on average (as in the data).

II.4.1B Classical Linear Regression: Models

To explore how r.v. Y depends on $X_1, \ldots, X_k$: assume

$$\begin{aligned} Y &= [\beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k] + \epsilon \\ \text{(response} &= [\text{ linear function of } X_1, \ldots, X_k] + \text{error)} \end{aligned}$$

With n indpt obs on Y, where Y_i is associated with the values $x_{1i}, \ldots, x_{ki}$ of $X_1, \ldots, X_k$ for $i = 1, \ldots, n$:

$$\begin{aligned} Y_1 &= \beta_0 + \beta_1 x_{11} + \ldots + \beta_k x_{1k} + \epsilon_1 \\ Y_2 &= \beta_0 + \beta_1 x_{21} + \ldots + \beta_k x_{2k} + \epsilon_2 \\ \vdots & \quad \vdots \\ Y_n &= \beta_0 + \beta_1 x_{n1} + \ldots + \beta_k x_{nk} + \epsilon_n \end{aligned}$$

where (1) $\epsilon_1, \ldots, \epsilon_n$ are indpt, and (2) $E(\epsilon_i) = 0$, $Var(\epsilon_i) = \sigma^2$.

That is,

$$\begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1k} \\ 1 & x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{nk} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_0 \\ \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_k \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

Classical Linear Regression Model

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

with (1) $E(\boldsymbol{\epsilon}) = \mathbf{0}$, and (2) $Var(\boldsymbol{\epsilon}) = E(\boldsymbol{\epsilon}\boldsymbol{\epsilon}') = \sigma^2\mathbf{I}$.

- the response vector Y: $\mathbf{Y}_{n\times 1}$
- the design matrix: $\mathbf{X}_{n\times(k+1)}$
- the parameter vector: $\boldsymbol{\beta}_{(k+1)\times 1}$
- the error vector: $\boldsymbol{\epsilon}_{n\times 1}$

Simple Linear Regression Model.
How Y depends on X_1 linearly?

- $E(Y|X_1 = x_1) = \beta_0 + \beta_1 x_1$. Equivalently, $Y = \beta_0 + \beta_1 X_1 + \epsilon$ with $E(\epsilon) = 0$ and $Var(\epsilon) = \sigma^2$.
- With n indpt obs on Y, $y_1, \ldots, y_n$ corresponding to $x_{11}, \ldots, x_{n1}$ (n values of X_1), $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ with $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $Var(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}$

Heights of Father & Son is an example of the Simple Linear Regression Model.

- One-way univariate ANOVA: how Y depends on X, a categorical variable?
  - **Example 7.2 (textbook p363)**: compare three population means: $n = 8$ for indpt obs on Y, where obs are from populations 1, 2, 3.

Suppose Y has three population means μ_1, μ_2, μ_3. The one-way ANOVA model:

$$Y_{lr} = \mu + \tau_l + \varepsilon_{lr}, \qquad l = 1, 2, 3,$$

where $\varepsilon_{lr} \sim N(0, \sigma^2)$ and $\sum_{l=1}^{3} \tau_l = 0$.

Can be written as regression model by introducing *dummy* variables:

$$X_1 = \mathbf{1}\{\text{population 1}\}, \quad X_2 = \mathbf{1}\{\text{population 2}\}, \quad X_3 = \mathbf{1}\{\text{population 3}\}.$$

Let $\beta_0 = \mu$ and $\beta_l = \tau_l$ for $l = 1, 2, 3$. Then

$$Y_j = \beta_0 + \beta_1 X_{1j} + \beta_2 X_{2j} + \beta_3 X_{3j} + \varepsilon_j, \qquad j = 1, \dots, n.$$

- Two-way univariate ANOVA: how Y depends on two categorical variables X and Z?

II.4.1C Classical Linear Regression: Least Squares Estimation (LSE)

Recall the linear regression model:

$$Y = \beta_0 + \beta_1 X_1 + \ldots + \beta_k X_k + \epsilon$$

with $E(\epsilon) = 0$ and $Var(\epsilon) = \sigma^2$. To estimate $\beta_0, \ldots, \beta_k$?

When there are n indpt obs on Y: $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ with (1) $E(\boldsymbol{\epsilon}) = \mathbf{0}$, and (2) $Var(\boldsymbol{\epsilon}) = E(\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{'}) = \sigma^2\mathbf{I}$.
How to estimate $\beta_0, \ldots, \beta_k$?

Least Squares Estimation. To select $\mathbf{b}$ such that

$$S(\mathbf{b}) = \sum_{i=1}^{n} \left(y_i - [b_0 + b_1 x_{i1} + \ldots + b_k x_{ik}]\right)^2 = (\mathbf{y} - \mathbf{Xb})^{'}(\mathbf{y} - \mathbf{Xb})$$

is minimized.

- Denote the selection by $\hat{\boldsymbol{\beta}}$, the LSE of $\boldsymbol{\beta}$.
- the *fitted* model: $Y = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \ldots + \hat{\beta}_k X_k$
- the *residuals*: $\hat{\epsilon}_i = y_i - \hat{y}_i = y_i - [\hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \ldots + \hat{\beta}_k x_{ik}]$

II.4.1C Classical Linear Regression: Least Squares Estimation (LSE)

Least Squares Estimator.

$$\hat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{Y}$$

provided $\mathbf{X}$ has full rank.

- the fitted values: $\hat{\mathbf{Y}} = \mathbf{X}\hat{\beta} = \mathbf{H}\mathbf{Y}$ with the "hat" matrix $\mathbf{H} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'$.
- the residuals: $\hat{\epsilon} = \mathbf{Y} - \hat{\mathbf{Y}} = (\mathbf{I} - \mathbf{H})\mathbf{Y}$, and $S(\hat{\beta}) = \hat{\epsilon}'\hat{\epsilon} = |\hat{\epsilon}|^2$.
- $\mathbf{X}'\hat{\epsilon} = \mathbf{0}$ and $\hat{\mathbf{y}}'\hat{\epsilon} = 0$.
- Example 7.4 in the textbook (page 372)

II.4.1C Classical Linear Regression: Least Squares Estimation (LSE)

- *Sum-of-Squares Decomposition.* $\mathbf{y}'\mathbf{y} = \hat{\mathbf{y}}'\hat{\mathbf{y}} + \hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}}$.
- *Sampling Properties of LSE.*
  - $E(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$ and $Var(\hat{\boldsymbol{\beta}}) = \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$
  - $E(\hat{\boldsymbol{\epsilon}}) = \mathbf{0}$ and $Var(\hat{\boldsymbol{\epsilon}}) = \sigma^2 [\mathbf{I} - \mathbf{H}]$
  - $E(\hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}}) = (n-k-1)\sigma^2$.
    Thus $s^2 = \frac{\hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}}}{n-[k+1]}$ with $E(s^2) = \sigma^2$.
  - **Gauss least squares theorem** The LSE $\hat{\boldsymbol{\beta}}$ is the best linear unbiased estimator of $\boldsymbol{\beta}$.

II.4.2 Linear Regression Based Inference (Chp7.4-5)

Consider $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$.

Inferential procedures under the normality assumption on the error term $\boldsymbol{\epsilon}$: $\boldsymbol{\epsilon} \sim MN(\mathbf{0}, \sigma^2\mathbf{I})$

- **II.4.2A** on the regression model (Chp7.4);
- **II.4.2B** by the fitted model (Chp7.5).

II.4.2A Linear Regression Based Inference

- The LSE $\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \sim MN(\boldsymbol{\beta}, \sigma^2(\mathbf{X}'\mathbf{X})^{-1})$. The same as the MLE.

- The MLE $\hat{\sigma}^2 = \hat{\boldsymbol{\epsilon}}'\hat{\boldsymbol{\epsilon}}/n$, and $n\hat{\sigma}^2/\sigma^2 \sim \chi^2_{n-[k+1]}$.

- A $100(1-\alpha)\%$ confidence region (ellipsoid) for $\boldsymbol{\beta}$ is

$$\left\{\boldsymbol{\beta} : (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})'\mathbf{X}'\mathbf{X}(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) \le (k+1)s^2 F_{k+1,n-[k+1]}(\alpha)\right\}$$

- Simultaneous $100(1-\alpha)\%$ confidence intervals for β_j, $j = 0, 1, \ldots, k$, are

$$\hat{\beta}_j \pm \sqrt{\widehat{Var}(\hat{\beta}_j)}\sqrt{(k+1)F_{k+1,n-[k+1]}(\alpha)}$$

Here $\widehat{Var}(\hat{\beta}_j)$ is the diagnoal element of $s^2(\mathbf{X}'\mathbf{X})^{-1}$ corresponding to $\hat{\beta}_j$.

Example. (textbook p372) [Linear regression analysis with real estate data] $n = 20$

- Y = selling price (in \$1000)
- Z_1 = total dwelling size (100 square feet); Z_2 = assessed value (in \$1000)
- Inference on regression coefficients? Interval estimates? Hypothesis test on $H_0 : \beta_j = 0$?

II.4.2A Linear Regression Based Inference

Consider

$$\boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{\beta}_{(1)} \\ \boldsymbol{\beta}_{(2)} \end{pmatrix}, \quad \mathbf{X} = \left(\mathbf{X}_1 \middle| \mathbf{X}_2\right)$$

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} = \mathbf{X}_1\boldsymbol{\beta}_{(1)} + \mathbf{X}_2\boldsymbol{\beta}_{(2)} + \boldsymbol{\epsilon}$$

To test $H_0 : \boldsymbol{\beta}_{(2)} = \mathbf{0}$ with $\boldsymbol{\beta}_{(2)}^{'} = (\beta_{q+1}, \ldots, \beta_k)$?

- Likelihood ratio test:

$$\frac{\max_{\boldsymbol{\beta}_{(1)}, \sigma^2} L(\boldsymbol{\beta}_{(1)}, \sigma^2)}{\max_{\boldsymbol{\beta}, \sigma^2} L(\boldsymbol{\beta}, \sigma^2)} = \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}^2}\right)^{-n/2} = \left(1 + \frac{\hat{\sigma}_1^2 - \hat{\sigma}^2}{\hat{\sigma}^2}\right)^{-n/2}$$

- It is equivalent to test H_0 : Model$_0$$\mathbf{Y} = \mathbf{X}_1\boldsymbol{\beta}_{(1)} + \boldsymbol{\epsilon}$ vs H_1 : Model$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$

$$F = \frac{[SS_{res}(Model_0) - SS_{res}(Model)]/(k-q)}{SS_{res}(Model)/(n-[k+1])} \sim F_{k-q, n-[k+1]}$$

under H_0

II.4.2B Inference by the Fitted Linear Regression Model

Provided the LSE $\hat{\boldsymbol{\beta}}$ using data with size n for the model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$ under the normality assumption $\boldsymbol{\epsilon} \sim MN(\mathbf{0}, \sigma^2\mathbf{I})$:
if Y_0 is the response when the indpendent variables are $x_{01}, \ldots, x_{0k}$,

- *to estimate* $E(Y_0|\mathbf{x}_0) = \beta_0 + \beta_1 x_{01} + \ldots + \beta_k x_{0k} = \mathbf{x}_0^{'}\boldsymbol{\beta}$
  - point estimate: $\mathbf{x}_0^{'}\hat{\boldsymbol{\beta}}$
  - $100(1-\alpha)\%$ confidence interval:

$$\mathbf{x}_0^{'}\hat{\boldsymbol{\beta}} \pm t_{n-[k+1]}(\alpha/2)\sqrt{[\mathbf{x}_0'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_0]s^2}$$

- *to predict for* $Y_0 = \mathbf{x}_0^{'}\boldsymbol{\beta} + \epsilon_0$
  - point estimate: $\mathbf{x}_0^{'}\hat{\boldsymbol{\beta}}$
  - $100(1-\alpha)\%$ prediction interval:

$$\mathbf{x}_0^{'}\hat{\boldsymbol{\beta}} \pm t_{n-[k+1]}(\alpha/2)\sqrt{[\mathbf{x}_0'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}_0 + 1]s^2}$$

What will we study next? - Guest Lecture

- *Part I. Introduction and Preparation*

- **Part II. Inference under Multivariate Normal Distribution (Chp 4-7)**
  - *II.1 Multivariate Normal Distribution (Chp 4)*
  - *II.2 Inferences on Mean Vector (Chp 5)*
  - *II.3 Comparisons of Several Mean Vectors (Chp 6)*

  - **II.4 Multivariate Linear Regression (Chp 7)**
    - *II.4.1 Classical Linear Regression (Chp7.1-3)*
    - *II.4.2 Linear Regression Based Inference (Chp7.4-5)*
    - **II.4.3 Model Checking (Chp7.6)**
    - **II.4.4 Multivariate Multiple Regression (Chp7.7)**

- *Part III. Commonly-Used Multivariate Analysis Methods (Chp 8-11)*