What to do today ?

Part I. Introduction and Preparation

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Part II.1.1 Multivariate Normal Distribution $MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (Chp 4.1-2)

The most important distribution in all of Statistics is the normal (Gaussian) distribution.

II.1.1A. Multivariate normal distribution Definition. A p-dim r.v. $\mathbf{X}$

has a *normal* distribution if its pdf

$$f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{|2\pi\boldsymbol{\Sigma}|}} \exp\Big\{ -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\Big\}, \quad -\infty < \mathbf{x} < \infty,$$

where $\boldsymbol{\Sigma}$ is positive definite. Denote it by $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

- If $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $E(\mathbf{X}) = \boldsymbol{\mu}$ and $V(\mathbf{X}) = \boldsymbol{\Sigma}$.
- $MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: a family of (multivariate) distributions. e.g. $MN_p(\mathbf{0}, \mathbf{I})$, *the standard (multivariate) normal distribution*.

More about the normal distributions ...

- The pdf is symmetric about $\boldsymbol{\mu}$.
- As $\boldsymbol{\mu}$ changes, the mode of the pdf curve shifts accordingly.
- As $\boldsymbol{\Sigma}$ changes, the spread and/or shape of the pdf surface change accordingly.

II.1.1B. Shape of Multivariate Normal Density

Suppose $\mathbf{X} \sim BN(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: its pdf

$$f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{\sqrt{|2\pi\boldsymbol{\Sigma}|}} \exp\left\{ -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}, \quad -\infty < \mathbf{x} < \infty,$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2)'$ and $\boldsymbol{\Sigma} = (\sigma_{ij}) = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$.

- If $\rho = 0$, $f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \phi(x_1; \mu_1, \sigma_1^2)\phi(x_2; \mu_2, \sigma_2^2)$: $\phi(x; \mu, \sigma^2)$ is the pdf of $X \sim N(\mu, \sigma^2)$.
- The density $f(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a constant for all $\mathbf{x} = (x_1, x_2)'$ satisfy $(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = c^2$. (It defines an ellipse centered at $\boldsymbol{\mu} = (\mu_1, \mu_2)'$.)

Normal density contour: $\left\{\mathbf{x} : (\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}) = c^2\right\}$

- It defines an ellipse centered at $\boldsymbol{\mu} = (\mu_1, \mu_2)'$.
- The ellipse's axes are $\pm c\sqrt{\lambda_j}\mathbf{e}_j$ for $j = 1, 2$, where $\lambda_j, \mathbf{e}_j$ are *eigenvalue* and *eigenvector* pairs of $\boldsymbol{\Sigma}$.

II.1.1C. Important Properties of Multivariate Normal Distribution

- If $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\mathbf{Y} = \mathbf{A}'\mathbf{X} + \mathbf{d} \sim MN_p(\mathbf{A}'\boldsymbol{\mu} + \mathbf{d}, \mathbf{A}'\boldsymbol{\Sigma}\mathbf{A})$.
  e.g. $\mathbf{Y} = \boldsymbol{\Sigma}^{-1/2}[\mathbf{X} - \boldsymbol{\mu}] \sim MN_p(\mathbf{0}, \mathbf{I})$.
- $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \iff \mathbf{A}'\mathbf{X} \sim MN_q(\mathbf{A}'\boldsymbol{\mu}, \mathbf{A}'\boldsymbol{\Sigma}\mathbf{A})$ for any $\mathbf{A} \in \mathcal{R}^{p\times q}$.
- If $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and
  $\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$,
  $\mathbf{X}_1 \sim MN_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$ and $\mathbf{X}_2 \sim MN_{p_2}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$.
- If $\mathbf{X}_1 \sim MN_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$ and $\mathbf{X}_2 \sim MN_{p_2}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$, then $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent $\iff$

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \sim MN_{p_1+p_2}\left(\begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{22} \end{pmatrix} \right).$$

- If $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ with $\boldsymbol{\Sigma}$ invertible, $(\mathbf{X} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X} - \boldsymbol{\mu}) \sim \chi^2_p$, the Chi-square distribution with degree of freedom p.

II.1.2 Estimation of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (Chp 4.3)

Consider $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: what are $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?
Suppose $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are iid observations on $\mathbf{X}$, how to use the data to estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?

II.1.2A. By Method of Moments

Review *univariate* MME: Consider $X \sim F(.; \theta_1, \ldots, \theta_m)$.
The **MME** $(\hat{\theta}_1, \ldots, \hat{\theta}_m)$ are the solution to the equations:

$$\hat{\mu}_k = \mu_k(\theta_1, \ldots, \theta_m), \text{k=1,...,m}$$

- kth population moment $\mu_k = \mu_k(\theta_1, \ldots, \theta_m) = E(X^k)$
- kth sample moment $\hat{\mu}_k = \frac{1}{n}(X_1^k + \ldots + X_n^k)$

e.g. $\hat{\mu} = \hat{\mu}_1 = \bar{X}$; $\hat{\sigma}^2 = \hat{\mu}_2 - (\hat{\mu}_1)^2$.

II.1.2 Estimation of μ and Σ (Chp 4.3)

Consider $\mathbf{X} \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: what are $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?
Suppose $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are iid observations on $\mathbf{X}$, how to use the data to estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$?

II.1.2A. By Method of Moments

- $E(\mathbf{X}) = \boldsymbol{\mu}$ and thus the MME $\hat{\boldsymbol{\mu}} = \bar{\mathbf{X}} = \sum_{i=1}^{n} \mathbf{X}_i \big/ n$, the sample mean.

- $V(\mathbf{X}) = \boldsymbol{\Sigma} = E(\mathbf{X}\mathbf{X}') - \big(E\mathbf{X}\big)\big(E\mathbf{X}\big)'$ and thus the MME

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i\mathbf{X}_i' - \bar{\mathbf{X}}\bar{\mathbf{X}}'.$$

$\hat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})' = \frac{n-1}{n}\mathbf{S}$.

II.1.2 Estimation of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (Chp 4.3)

II.1.2B. By Maximum Likelihood Estimation

The likelihood function is

$$
\begin{aligned}
L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) &= \prod_{i=1}^{n} f(\mathbf{x}_i; \boldsymbol{\mu}, \boldsymbol{\Sigma}) \\
&= \prod_{i=1}^{n} \frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left[-\frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x}_i - \boldsymbol{\mu})' \right].
\end{aligned}
$$

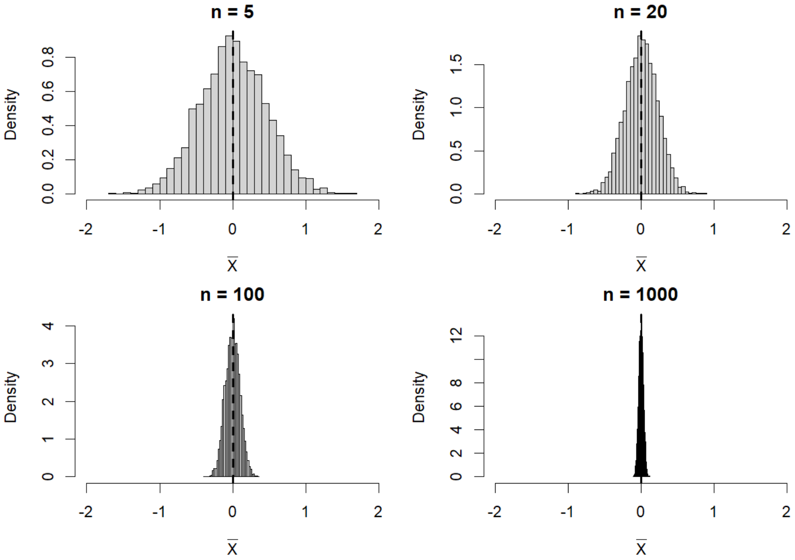
Maximizing $L(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ wrt $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ yields the MLE:

$$
\hat{\boldsymbol{\mu}} = \bar{\mathbf{X}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{X}_i, \hat{\boldsymbol{\Sigma}} = \frac{1}{n} \sum_{i=1}^{n} (\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})' = \frac{n-1}{n} \mathbf{S}.
$$

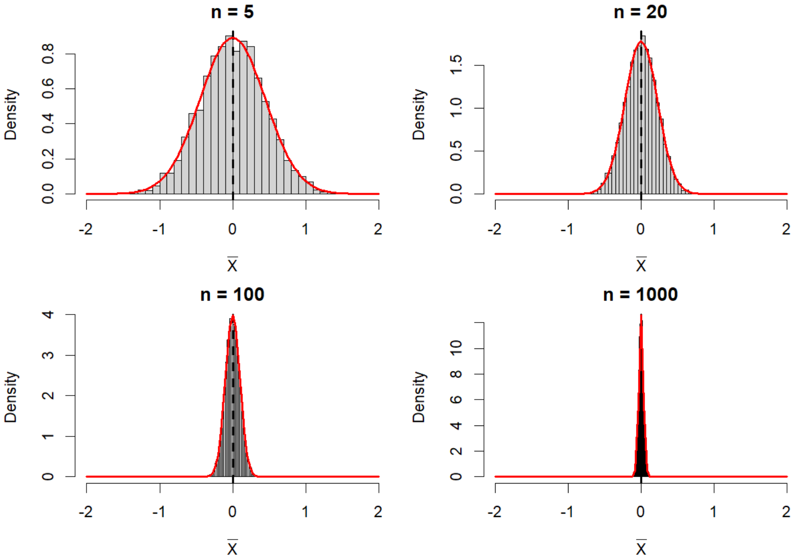
The *MLE* of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the same as the *MME*.

II.1.3 Properties of $\bar{\mathrm{X}}$ and S (Chp 4.4-5)

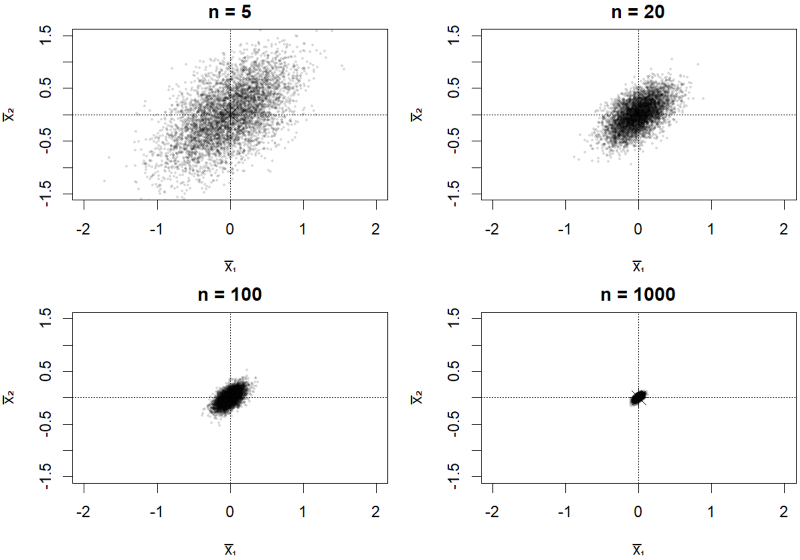
- *Unbiased estimators*: $E(\bar{\mathbf{X}}) = \boldsymbol{\mu}$ and $E(\mathbf{S}) = \boldsymbol{\Sigma}$, while $E(\hat{\boldsymbol{\Sigma}}) = \frac{n-1}{n}\boldsymbol{\Sigma}$.
- *Relationship*: $\bar{\mathbf{X}}$ and $\mathbf{S}$ are independent; so are $\bar{\mathbf{X}}$ and $\hat{\boldsymbol{\Sigma}}$.
- *Limits:* $\bar{\mathbf{X}} \to \boldsymbol{\mu}$ and $\mathbf{S} \to \boldsymbol{\Sigma}$ as $n \to \infty$.
- *Distributions:*
  - $\bar{\mathbf{X}} = \sum_{i=1}^{n} \mathbf{X}_i / n \sim MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}/n)$ by noting
    $$\bar{\mathbf{X}} = \frac{1}{n}(\mathbf{I}, \ldots, \mathbf{I}) \begin{pmatrix} \mathbf{X}_1 \\ \vdots \\ \mathbf{X}_n \end{pmatrix}.$$



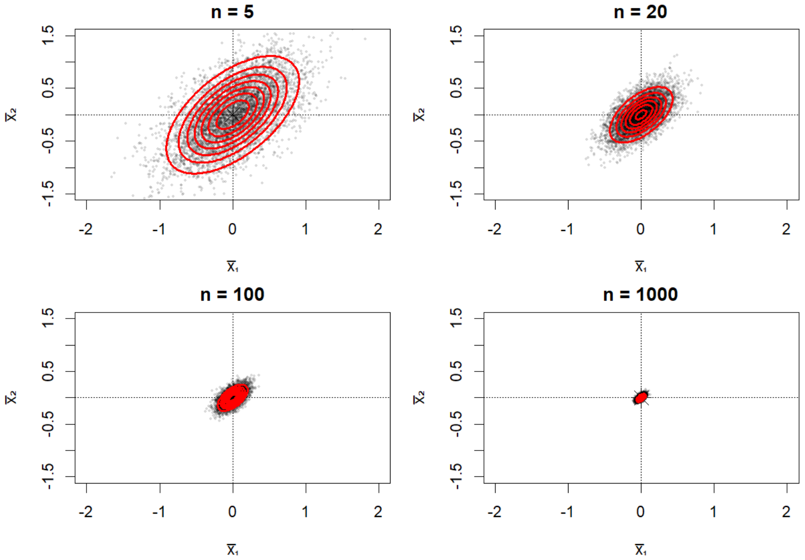
Example: $\bar{X}$ of uni-variate normal distribution.



Example: $\bar{X}$ of uni-variate normal distribution.



Example: $\bar{\mathbf{X}}$ of bi-variate normal distribution.



Example: $\bar{\mathbf{X}}$ of bi-variate normal distribution.

Distributions:

- $n\hat{\mathbf{\Sigma}} = (n-1)\mathbf{S} = \sum_{i=1}^{n}(\mathbf{X}_i - \bar{\mathbf{X}})(\mathbf{X}_i - \bar{\mathbf{X}})'$ follows a *Wishart* distribution with degree of freedom $n-1$.

Definition. If $\mathbf{Z}_1, \ldots, \mathbf{Z}_m$ are indpt and follow $MN_p(\mathbf{0}, \mathbf{\Sigma})$, the distribution of $\sum_{j=1}^{m} \mathbf{Z}_j \mathbf{Z}_j'$ is the Wishart Distribution $W_p(\mathbf{\Sigma}, m)$.

- Special case of $p = 1$ and $\mathbf{\Sigma} = \sigma^2 = 1$: χ^2_m-distn

- Suppose $\mathbf{W}_1 \sim W_p(\mathbf{\Sigma}, m_1)$ and $\mathbf{W}_2 \sim W_p(\mathbf{\Sigma}, m_2)$. If $\mathbf{W}_1$ and $\mathbf{W}_2$ are indpt, $\mathbf{W}_1 + \mathbf{W}_2 \sim W_p(\mathbf{\Sigma}, m_1 + m_2)$

- If $\mathbf{W} \sim W_p(\mathbf{\Sigma}, m)$, $\mathbf{CWC}' \sim W_q(\mathbf{C\Sigma C}', m)$ when $\mathbf{C}$ is $q \times p$ matrix with rank of q.

What will we study next?

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- **Part II. Inference under Multivariate Normal Distribution (Chp 4-7)**
  - **II.1 Multivariate Normal Distribution (Chp 4)**
    - *II.1.1 Multivariate Normal Distribution $MN_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (Chp 4.1-2)*
    - *II.1.2 Estimation of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ (Chp 4.3)*
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  - *II.2 Inferences on Mean Vector (Chp 5)*
  - *II.3 Comparisons of Several Mean Vector (Chp 6)*
  - *II.4 Multivariate Linear Regression (Chp 7)*
- *Part III. Commonly-Used Multivariate Analysis Methods (Chp 8-11)*
- *Part IV. Other Topics (Chp 12)*